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Sedimentation and sediment flow on inclined surfaces

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The steady sedimentation of a suspension over an inclined surface is analysed by considering the combined effects of settling hindrance, bulk motion and particle resuspension. The coupled momentum and mass balances suggest that a thin high-density sediment layer will form over the inclined surface, reminiscent of the thin thermal boundary layers in the classical problem of natural convection. It is shown that for a given value of the particle volume fraction in the unsettled suspension, a steady flow of the sediment can be maintained only if the angle of inclination exceeds a minimum value. The analysis further predicts the existence of a sharp discontinuity in the particle volume fraction across the suspension–sediment interface along which the bulk velocity has a local maximum. High particle volume fractions within the sediment are predicted when the unsettled suspension is either very dilute or very concentrated. This leads to the formation of relatively large sediment-layer thicknesses which reflect the fact that a large body force is required in these two limiting cases to overcome the viscous resistance to flow near the inclined boundary.

1. Introduction

The thickening of slurries and the separation between a particulate phase and the continuous fluid in which it is dispersed constitute one of the most common operations in industrial processes. Often, this is achieved using gravity settling, which is a relatively slow process when the particles are small and the fluid is viscous. Settlers are, therefore, large pieces of equipment which typically occupy a considerable area and hence it is desirable to be able to scale down the size of settling tanks without reducing their operational capacity. One way of achieving such an enhancement in performance is by using devices which take advantage of the Boycott effect (Boycott 1920). Specifically, these vessels are equipped with a series of closely spaced inclined parallel plates which, as was pointed out by Ponder (1925) and by Nakamura & Kuroda (1937), greatly enhance the surface area for capturing the sedimenting particles with the result that the suspension is separated in a time much shorter than that required when the inclined surfaces are absent. This explanation for the enhancement, although kinematically correct in most practical cases, cannot account, however, for the convective currents which are observed along the underside of the inclined surfaces when the particles are heavier than the fluid or for the occurrence of waves along the suspension/clear-fluid interface. Hence the sedimentation between two inclined surfaces has become a fluid-mechanical phenomenon of both fundamental and practical interest.

During the past decade, a comprehensive theory has been developed based on the principles of continuum mechanics, which has succeeded in describing quantitatively the motion within the clear fluid layer and the adjoining suspension (the latter treated as an effective Newtonian fluid) for a wide range of parameters and operating conditions. The principal features of this theory as well as many of the major results are presented in the review by Davis & Acrivos (1985) and in the recent article by Borhan & Acrivos (1988) which also contain references to the earlier studies by a number of other investigators.

The effectiveness of the entire sedimentation process also depends, however, on the dynamics of the sediment layer containing the settled particles. Specifically, the question of whether this heavy layer will remain stagnant and simply continue to grow in thickness along the top side of the inclined surface, or whether it will slide along freely, is interesting in itself but also has obviously practical implications since the conditions under which such a flow can exist will dictate the feasibility of operating such devices under steady-state conditions. Yet, this question has received little attention to date because, although a flowing sediment layer can often be observed in laboratory experiments, no known mechanism that could negate the tendency of the sedimentation process to compress the sediment layer and would, thereby, sustain such a motion had been recognized. Consequently, either the influence of this sediment layer was neglected or, in the rare instances when it was taken into account (cf. Probstein, Yung & Hicks 1981; Leung & Probstein 1983), it was treated as an effective fluid of *a priori* specified composition.

Recently, a number of experimental studies (Gadala-Maria & Acrivos 1980; Leighton & Acrivos 1986, 1987*a*, *b*) have led to the conclusion, however, that various curious phenomena which had been observed in sheared suspensions with high solid volume fractions, for example the slow variation in the measured effective viscosity in a Couette viscometer and the resuspension of a settled bed of particles into a clear fluid flowing above it under shear, were due to shear induced particle migration across bulk streamlines. This occurs because the motion of an individual particle is affected by its hydrodynamic interaction with the numerous particles surrounding it which also interact with each other, with the result that the complex pattern of interactions combines to create a shear-induced diffusion and therefore particle migration that in turn affects the particle concentration distribution. Computer simulations of Stokesian dynamics (cf. Brady & Bossis 1988; Durlofsky & Brady 1989) have also supported the above observations and confirmed that even in the absence of inertia effects a settled bed of particles will resuspend into a flowing fluid stream undergoing shear.

Clearly, when a suspension in which particles are settling is in motion, the particle concentration is determined by a combination of the sedimentation rate, the bulk flow and the shear-induced diffusion. This distribution in turn influences the bulk properties such as the effective viscosity and density, thereby strongly affecting the bulk motion and the local value of the shear rate which induces particle migration. Thus, it would appear that the prediction of the motion of such concentrated suspensions should take into account the strong coupling that exists between the bulk characteristics of the flow and the migration of the settling particles on the microscale. In fact, using this approach, Schaflinger, Acrivos & Zhang (1989) have recently studied the forced flow in a channel and the gravity-driven film flow over an inclined surface of a suspension of settling particles, and by employing the resuspension model of Leighton & Acrivos (1986) established conditions under which such flows can exist in a steady state, i.e. the conditions under which the shear rate distribution in the flowing suspension induces a resuspension which is sufficient to prevent the particles from settling further.

In this communication we shall present an analysis of the gravity-driven sedimentation of particles over an inclined surface and thereby address an aspect of particle settling in inclined vessels which, to date, appears to have been neglected. We shall anticipate that the heavy sediment layer will slide along the inclined surface in a manner similar to the natural convective motion of a heated or cooled fluid layer, and shall seek to determine the conditions under which this motion can exist in a steady state. When this occurs, the combined effect of the settling rate and the shear-induced particle resuspension that is driven by the bulk motion of the sediment, creates a particle concentration distribution which sustains the macroscopic flow and prevents an accumulation of the sediment along the inclined surface. The general formulation of the problem is given in §2, while in §3 we examine a two-dimensional flow over an inclined flat plane. A similarity-type solution is developed in §4 for the case of a highly viscous sediment layer within which bulk inertia effects can be neglected. The results of this calculation are presented and discussed in §5 which focuses on the conditions for the existence of a steady-state solution to the relevant equations alluded to above. It is shown that, as expected, the physical parameters that may limit the range over which this steady flow can exist are the particle volume fraction in the bulk of the suspension and the angle of inclination. Questions regarding the stability of the motion of the sediment layer and the possible occurrence of secondary flows are not addressed and are left for further investigation.

2. Formulation

Consider an effectively infinite body of viscous fluid containing a dispersion of small solid particles which are sedimenting due to gravity. The particles are taken to be spherical, uniform in size with radius a and sufficiently small that inertia effects on the microscale are negligible. The settling particles encounter a surface of length L which is inclined at an angle α as depicted in figure 1. The solid volume fraction far from the surface is ϕ_s. We treat the suspension and the sediment layer as effective Newtonian fluids whose effective properties depend only on the local particle volume fraction, ϕ. Thus the effective density and viscosity are given by

$$\rho(\phi) = \rho_f \gamma(\phi) = \rho_f \left[1 + \frac{\Delta\rho}{\rho_f}\phi\right] \tag{2.1}$$

and

$$\mu(\phi) = \mu_f \lambda(\phi), \tag{2.2}$$

where ρ_f and μ_f are the density and the viscosity of the clear fluid, respectively, $\Delta\rho = \rho_s - \rho_f$ with ρ_s being the density of the particles, and $\lambda(\phi)$ denotes the effective relative viscosity. In addition, we suppose that the particle slip velocity relative to the bulk is

$$\boldsymbol{u}^* = u_t f(\phi)\, \boldsymbol{g}/g \tag{2.3}$$

where $f(\phi)$ is the so-called hindrance function and u_t is the Stokes terminal sedimentation velocity for an isolated particle in a dilute suspension, i.e.

$$u_t = \frac{2}{9}\frac{g a^2 \Delta\rho}{\mu_f}, \tag{2.4}$$

with $g = |\boldsymbol{g}|$ being the gravitational acceleration. On the other hand, the shear-induced particle diffusion coefficient depends on both the local volume fraction and

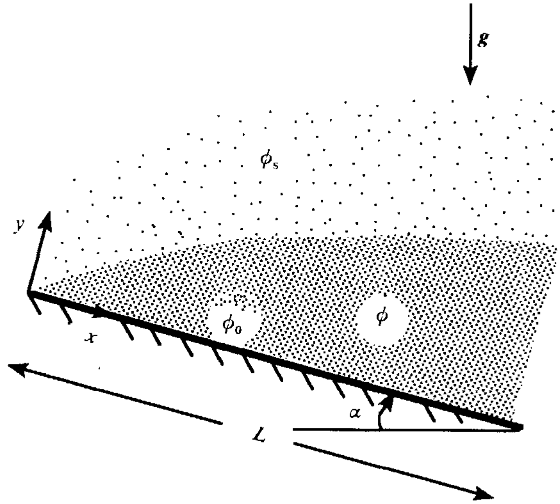

FIGURE 1. A schematic description of sedimentation and sediment flow on an inclined surface.

on the local flow field but dimensional analysis suggests that it should be linear in the bulk velocity gradient and hence, assuming that, to a first approximation, the diffusivity is isotropic, we express the shear-induced diffusion coefficient as (Leighton & Acrivos 1987*a, b*)

$$D(\phi) = |\nabla \boldsymbol{u}|\, a^2 \beta(\phi), \tag{2.5}$$

where $|\nabla \boldsymbol{u}|$ is the magnitude of the local gradient of the bulk velocity field, $\boldsymbol{u}$, and β is a function which is assumed to depend only on ϕ. The non-dimensional functions λ, f and β are taken to be known and the explicit forms used in this analysis will be discussed in §5.

2.1. *Basic equations*

The momentum balance for the steady motion of the suspension has the familiar form

$$\rho(\phi)\, \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\cdot\boldsymbol{\sigma} + (\rho(\phi)-\rho_f)\,\boldsymbol{g}, \tag{2.6}$$

with

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu(\phi)\,(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\dagger}), \tag{2.7}$$

where $\boldsymbol{u}$, σ and p are the bulk velocity, stress and pressure fields, respectively. Also, the last term in (2.6) denotes the effective body force, proportional to the local density difference, which generates the anticipated motion. In addition. in view of the incompressibility condition, $\nabla\cdot\boldsymbol{u} = 0$.

The steady-state particle mass balance is given by

$$\boldsymbol{u}\cdot\nabla\phi = -\nabla\cdot\boldsymbol{J}, \tag{2.8}$$

where the total flux $\boldsymbol{J}$ is a combination of the dispersive flux due to the 'shear-induced diffusion effect' and of the sedimentation flux due to gravity. Hence

$$\boldsymbol{J} = -D\nabla\phi + \boldsymbol{u}^*\phi. \tag{2.9}$$

Equations (2.6) and (2.8) can be rendered dimensionless using L and u_t as the characteristic length and velocity respectively. Thus, incorporating the constant hydrostatic pressure associated with $\Delta\rho\phi_S g$ into the stress field, the momentum and mass balances become

$$R\gamma(\phi)\, \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\cdot\boldsymbol{\sigma} + \frac{L^2 g}{\mu_f u_t}(\phi-\phi_s)\frac{\boldsymbol{g}}{g}\Delta\rho \tag{2.10}$$

and

$$\boldsymbol{u}\cdot\nabla\phi = \frac{1}{Lu_t}\nabla\cdot(D\nabla\phi) - \nabla\cdot\left(f(\phi)\frac{\boldsymbol{g}}{g}\phi\right), \tag{2.11}$$

with $\boldsymbol{u}$, $\boldsymbol{\sigma}$ and the position vector now denoting dimensionless variables and $R = \rho_f u_t L/\mu_f$ being a macroscopic Reynolds number. The second dimensionless group in (2.10), having the familiar form of a Grashof number, can be further reduced using (2.4) to yield

$$\frac{gL^2\Delta\rho}{\mu_f u_t} = \frac{9}{2}\left(\frac{L^2}{a^2}\right) \equiv \tfrac{9}{2}\epsilon^{-2}, \tag{2.12}$$

which is clearly very large in most practical cases.

2.2. *Boundary conditions*

At large distances from the inclined surface we suppose that there is no bulk motion within the suspension and that the particle volume fraction is constant. Hence, the conditions

$$\boldsymbol{u} = 0, \quad p = 0, \quad \phi = \phi_s \tag{2.13}$$

apply.

At the inclined surface we invoke the no-slip boundary condition and also require that, at steady state, the particle flux normal to the surface should vanish. Thus

$$\boldsymbol{u} = 0, \quad \left(\frac{D}{Lu_t}\nabla\phi - f\frac{\boldsymbol{g}}{g}\phi\right)\cdot\boldsymbol{n} = 0, \tag{2.14}$$

where $\boldsymbol{n}$ is a unit vector normal to the surface. It is evident that, for a steady state to exist, the flux of settling particles must be balanced by a diffusive flux which is induced by the shearing motion of the suspension in the vicinity of the surface.

3. A two-dimensional case

In the two-dimensional geometry depicted in figure 1 we consider sedimentation over an inclined plate and neglect any changes normal to the (x, y)-plane. We first seek to simplify (2.10) and (2.11) by taking advantage of the fact that $\epsilon \ll 1$ in all cases of practical interest. Clearly, (2.10) is dominated by the buoyancy term which, in view of (2.12), is $O(\epsilon^{-2})$ and which must be balanced, at least in the vicinity of the solid wall, by viscous stresses. In addition, the convection and diffusive terms of (2.11) are of comparable magnitude. It is then easy to show that the appropriate scaling transformation is

$$Y = \epsilon^{-\frac{2}{3}}y, \quad u = \epsilon^{\frac{2}{3}}u_x, \quad v = u_y, \quad P = \epsilon^2 p, \quad \text{with} \quad \epsilon \equiv a/L \ll 1, \tag{3.1}$$

where u_x and u_y denote the velocity components in the x- and y-directions respectively. In view of the small value of ϵ, these transformations imply that, in the gravity-induced flow, the velocity components along the inclined surface dominate those perpendicular to it and that changes across the layer adjacent to the inclined surface are much steeper than those along it. Consequently, to a first approximation, the diffusivity is proportional to the component $|du/dy|$ of the velocity gradient tensor, and hence, in view of (3.1) and (2.5), we obtain for the particle mass balance equation

$$u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial Y} - \frac{\partial}{\partial Y}[\phi f(\phi)\cos\alpha] = \frac{\partial}{\partial Y}\left[\left|\frac{\partial u}{\partial Y}\right|\beta(\phi)\frac{\partial\phi}{\partial Y}\right] + O(\epsilon^{\frac{2}{3}}), \tag{3.2}$$

while, on using (2.2), we have for the components of (2.10)

$$R\gamma(\phi)\,\epsilon^{\frac{2}{3}}\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial Y}\right] = \tfrac{9}{2}(\phi-\phi_s)\sin\alpha+\frac{\partial}{\partial Y}\left(\lambda(\phi)\frac{\partial u}{\partial Y}\right)+O(\epsilon^{\frac{2}{3}}) \tag{3.3}$$

and

$$\frac{\partial P}{\partial Y} = O(\epsilon^{\frac{2}{3}}). \tag{3.4}$$

The condition at $Y = 0$, i.e. the zero flux requirement in (2.14), becomes

$$\beta(\phi)\left|\frac{\partial u}{\partial Y}\right|\frac{\partial \phi}{\partial Y}+\phi f(\phi)\cos\alpha = 0. \tag{3.5}$$

Equations (3.2) and (3.3) imply that the gravity-induced flow and the changes in the particle volume fraction occur within a thin boundary layer which exists in the vicinity of the inclined surface. Thus, if a steady state is to prevail, the particles that enter this layer from the bulk of the suspension must be carried downstream by the velocity component parallel to the surface following their redistribution due to shear-induced diffusion. The motion of the layer itself is driven by the x-component of the body force term in (3.3) and is opposed by the viscous shear stress. The relative importance of the inertial terms in (3.3) depends on the magnitude of $R\gamma(\phi)\,\epsilon^{\frac{2}{3}}$.

There is of course an obvious analogy between (3.2) and (3.3) and the corresponding equations for the natural convection of a viscous fluid near a heated inclined surface. Yet, fundamental differences appear in the boundary conditions. Specifically, in the thermally driven natural convection, the source of the density variation, i.e. the heat flux, penetrates the layer from the solid boundary side, and from there the heat propagates into the fluid by diffusion. It follows that the fluid motion can influence and regulate this heat flux through temperature gradients at the solid surface. In the present case, however, the source for the convection is located far from the inclined plate in a relatively quiescent fluid and, furthermore, the steady particle flux entering the boundary layer is entirely dominated by sedimentation and is not affected by diffusion.

To further simplify the analysis consider the case $R\gamma(\phi)\,\epsilon^{\frac{2}{3}} \ll 1$. The analogy to boundary-layer theory for thermal natural convection suggests then that, in the present case, the boundary layer could be viewed as consisting of two overlapping regions. In the region adjacent to the solid surface, which we shall term the viscous layer, inertia effects can be neglected and the gravitational force is balanced by the viscous stresses. Also, the longitudinal velocity component increases monotonically from zero at the solid surface to its maximum value at the edge of this region within which the entire variation of the particle volume fraction is confined. Separating the viscous layer and the quiescent suspension is a second sublayer in which the particle volume fraction has a constant value, i.e. $\phi = \phi_s$, and the momentum equation there reduces to a balance between the viscous and the inertia terms. But since the flow within this layer does not affect that in the sediment when $\epsilon \ll 1$, its structure will not be considered any further.

The two layers must satisfy matching conditions which, in this case, require that the velocity components and the particle flux be continuous across their common boundary. Moreover, since the longitudinal velocity components have identical scalings within these two regions, it follows from (3.1) and (3.3) that the thickness of the second layer relative to L is of the order of $R^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}$.

4. A similarity solution

As discussed above, when $Re^{\frac{2}{3}} \ll 1$, the inertia terms become negligible within the viscous layer. In addition, though, since the system of equations (3.2)–(3.5) together with the remaining boundary conditions no longer contains a characteristic lengthscale, it is natural to anticipate the existence of a similarity solution. Indeed it is easy to show that the appropriate transformation is

$$u = x^{\frac{2}{3}} F'(\eta) \cos\alpha, \quad \phi = \phi(\eta), \tag{4.1}$$

where

$$\eta = Y/x^{\frac{1}{3}}, \tag{4.2}$$

the prime denoting differentiation with respect to η. The use of the continuity equation readily yields $v = ((\frac{1}{3}\eta F') - F)\cos\alpha$, and substituting the above in (3.3) and (3.2) results in

$$(\lambda(\phi) F'')' + \tfrac{9}{2}(\phi - \phi_s)\tan\alpha = 0 \tag{4.3}$$

and

$$(\beta(\phi)\,|F''|\,\phi')' + F\phi' + \phi' \frac{\mathrm{d}}{\mathrm{d}\phi}(\phi f(\phi)) = 0. \tag{4.4}$$

The boundary conditions for (4.3) are

$$F = F' = 0 \quad \text{at} \quad \eta = 0 \tag{4.5}$$

and

$$F'' = 0 \quad \text{at} \quad \eta = \delta, \tag{4.6}$$

with $\eta = \delta$ denoting the outer edge of the viscous layer where the matching conditions with the outer region should apply. On the other hand, the condition for the particle balance at the inclined surface becomes

$$\beta(\phi)\,|F''|\,\phi' + f\phi = 0 \quad \text{at} \quad \eta = 0, \tag{4.7}$$

while at $\eta = \delta$ we require that the flux of particles leaving the inertia sublayer should equal that entering the viscous layer.

The flux matching condition deserves some further attention. Since the equation for the lines η = constant is

$$y - \epsilon^{\frac{2}{3}}\eta x^{\frac{1}{3}} = 0, \tag{4.8}$$

the components of a unit vector $\boldsymbol{n}$ normal to these lines are

$$n_x = -\frac{\epsilon^{\frac{2}{3}}\eta}{3x^{\frac{2}{3}}}\left(1 + \frac{\epsilon^{\frac{4}{3}}\eta^2}{9x^{\frac{4}{3}}}\right)^{-\frac{1}{2}}, \quad n_y = \left(1 + \frac{\epsilon^{\frac{4}{3}}\eta^2}{9x^{\frac{4}{3}}}\right)^{-\frac{1}{2}}. \tag{4.9}$$

Hence, in terms of the boundary-layer variables, the component of the velocity normal to η = constant, except near the origin $x = 0$, is $-F\cos\alpha$ and the total particle flux across these lines equals $-(F + f(\phi))\,\phi\cos\alpha$. Let the solids volume fraction in the viscous layer just below $\eta = \delta$ be denoted by ϕ_δ. Since the volume fraction above $\eta = \delta$ is $\phi = \phi_s$, the continuity condition of particle flux across $\eta = \delta$ attains the form

$$\phi_\delta(F_\delta + f(\phi_\delta)) = \phi_s(F_\delta + f(\phi_s)), \tag{4.10}$$

with F_δ denoting the value of F at $\eta = \delta$. Note that the left-hand side of (4.10) does not contain a term accounting for the diffusive flux of particles because, in view of (4.6), the coefficient of shear-induced diffusion vanishes at $\eta = \delta$.

An obvious solution to (4.10) is $\phi_\delta = \phi_s$. However, it is not too difficult to show that for any choice of the viscous-layer thickness δ (finite or infinite) the system of equations (4.3) and (4.4) subject to (4.5)–(4.7) and (4.10) with $\phi_\delta = \phi_s$ does not have

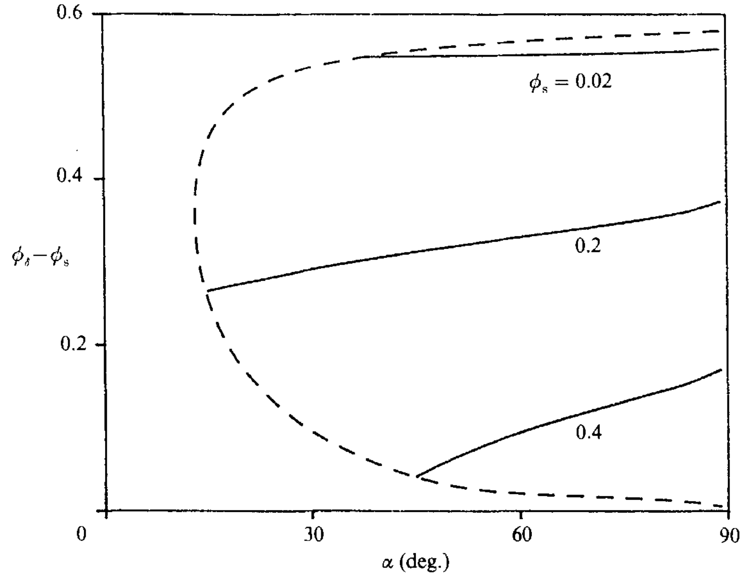

FIGURE 2. The jump in the particle volume fraction across the edge of the viscous boundary layer. The dotted curve denotes the boundary within which a steady-state solution exists.

a solution with ϕ everywhere finite. In other words (4.4) and the condition (4.6) are not compatible when $\phi_\delta = \phi_s$ for any choice of δ. A different solution to the flux continuity condition (4.10) must therefore be considered.

The existence of a dual solution to (4.10) is quite easily perceived if one notices that, when F_δ is not too large, the curve $\phi(F_\delta + f(\phi))$ has a maximum in the physically permitted range of ϕ since $f(\phi)$ is a monotonically decreasing function of ϕ. For example, in the absence of a bulk flow ($F_\delta = 0$), it is clear that the zero flux condition is satisfied for both $\phi = 0$ and $\phi = \phi_m$, where ϕ_m denotes the zero of $f(\phi)$, and that it is possible to have any given finite flux below some maximum at two different volume fractions. We conclude therefore that a jump in particle volume fraction across the viscous-layer boundary, $\eta = \delta$, is permissible.

To determine the value of this jump, we next examine (4.3) and (4.4) as $\eta \to \delta$ from below. We find that, in view of (4.6),

$$\left\{ F + \frac{d}{d\phi}(\phi f) - \frac{9\beta(\phi)}{2\lambda(\phi)}(\phi - \phi_s)\tan\alpha \right\}_{\phi_\delta} \phi' = 0 \quad \text{at} \quad \eta = \delta. \tag{4.11}$$

But since $\phi' \neq 0$, for otherwise $\phi = \phi_\delta$ throughout the sediment layer thereby violating (4.7), we conclude that the term in brackets should vanish.

The conditions which allow us to compute δ then become

$$F'' = 0, \quad \phi = \phi_\delta, \quad F = F_\delta \quad \text{at} \quad \eta = \delta, \tag{4.12}$$

where ϕ_δ and F_δ are uniquely determined by the simultaneous solution of (4.10) and (4.11). Figure 2 shows the jump in ϕ across the interface between the viscous layer and the layer above it for various angles of inclination. These values are calculated for a particular choice of the effective properties, $\lambda(\phi)$, $f(\phi)$ and $\beta(\phi)$, which will be described in §5. An interesting observation is that, for a suspension with low initial volume fraction, the minimum concentration within the sediment layer, i.e. the concentration ϕ_δ, is already relatively high and close to ϕ_m, the maximum permissible value of ϕ for all angles of inclination α. At higher values of ϕ_s, ϕ_δ is found to decrease with decreasing α. This, as we shall describe in the next section, has important implications with respect to the velocity profile and the layer thickness in these cases.

5. Results and discussion

A solution to (4.3) and (4.4) subject to (4.5), (4.7) and (4.12) requires the use of explicit expressions for the effective properties $\lambda(\phi)$, $f(\phi)$ and $\beta(\phi)$. Following Leighton & Acrivos (1986) we assume for the diffusivity the expression

$$\beta(\phi) = \tfrac{1}{3}\phi^2(1+\tfrac{1}{2}e^{8.8\phi}), \tag{5.1}$$

which is an empirical approximation fitted to the diffusivity data measured by Leighton (1985). The relative effective viscosity is similarly assumed to be of the form (Leighton & Acrivos 1987*b*):

$$\lambda(\phi) = \left[1+\frac{1.5\phi}{1-\phi/\phi_m}\right]^2, \quad \phi_m = 0.58, \tag{5.2}$$

where ϕ_m indicates the maximum possible particle volume fraction in the flowing suspension. Finally, the sedimentation hindrance function is obtained by supposing that a particle settles in an effective homogeneous medium with viscosity $\mu_f \lambda(\phi)$ and density $\rho_f \gamma(\phi)$. Thus, the correction factor becomes

$$f(\phi) = \frac{1-\phi}{\lambda(\phi)}, \tag{5.3}$$

which, although it does not conform to Batchelor's (1972) exact result $f \rightarrow 1-6.55\phi+O(\phi^2)$, is quite adequate for our purpose. Of course the precise forms selected for these effective parameters only affects the quantitative but not the qualitative features of the results.

A numerical solution of the fifth-order nonlinear boundary-value problem involves either iterations on a straightforward matrix inversion of a linearized problem which may not always converge, or the use of a shooting-method algorithm which requires the guess of three conditions and thus becomes a 3-parameter-space search. In both cases the calculations are tedious and therefore of limited usefulness, specifically considering the approximate nature of our model. On the other hand, one notices that ϕ and F are well-behaved monotonic functions of η which have smooth second and third derivatives, respectively, and which may well be approximated by the use of suitable polynomials. Hence, an approximate solution may be constructed reminiscent of the Kármán–Pohlhausen technique (Schlichting 1968) for classical boundary-layer flows. This approximation is described in the Appendix. We note that results which we obtained by direct numerical integration of the equations and by the means of the approximate method just mentioned were found to differ only very slightly over the entire range of the physical parameters considered.

Figure 3 depicts the particle volume fraction at the solid boundary, ϕ_0, and that at the edge of the viscous layer, ϕ_δ, as functions of ϕ_s, the volume fraction in the bulk suspension, for various angles of inclination α. Figure 4 shows the corresponding concentration profiles for $\alpha = 45°$ and for several values of ϕ_s. It is seen that, when ϕ_s is small ($\phi_s = 0.02$), the concentration of particles throughout the entire layer has a high value whereas, when ϕ_s is large ($\phi_s = 0.4$), ϕ is large almost everywhere except near $\eta = \delta$ where it equals the corresponding value of ϕ_δ. Calculations show that in

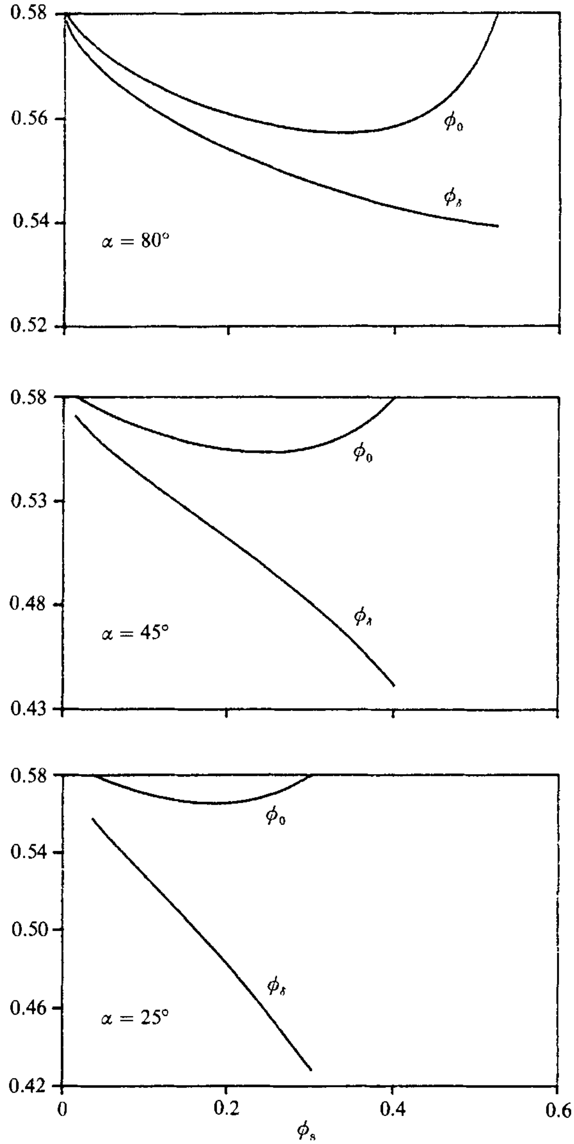

FIGURE 3. Particle volume fraction at the edge of the viscous layer, ϕ_δ, and at the inclined surface, ϕ_0, as a function of ϕ_s for various angles of inclination α.

these two extreme cases the average effective viscosity across the layer is over two orders of magnitude larger than that for the intermediate regimes ($\phi_s = 0.2$ and 0.3). Thus, according to our analysis, the sediment layer would be expected to flow under its own gravitational force more readily in the intermediate cases than in the extreme situations where a considerable body of heavy sediment is needed to maintain a flow in the presence of the high viscous resistance.

Indeed, the profiles of F', shown in figure 5 indicate that the longitudinal velocity along the edge of the sediment layer attains a maximum for some intermediate value of ϕ_s and diminishes when ϕ_s becomes larger or smaller. Similarly, the boundary-layer thickness parameter δ, which remains fairly constant over most of the permissible range of ϕ_s and especially at large angles of inclination, increases sharply

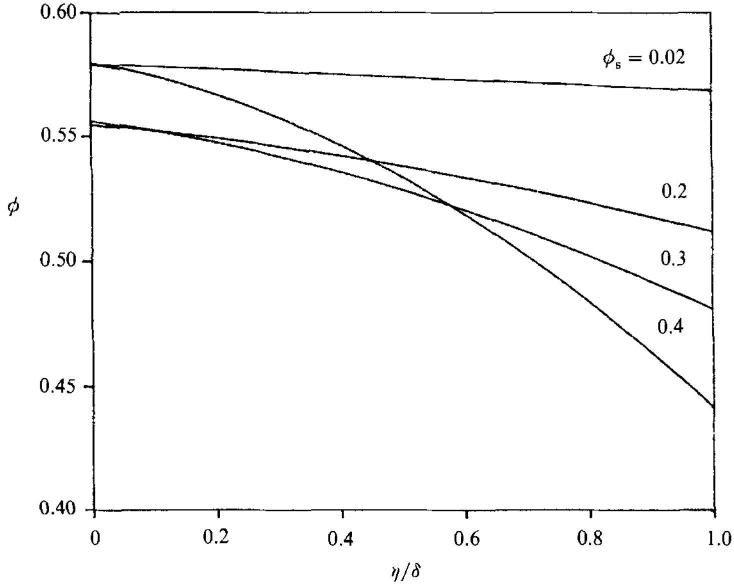

FIGURE 4. Particle volume fraction profiles across the viscous boundary layer for $\alpha = 45°$.

when ϕ_s becomes large or small (figure 6). An estimate for δ, in cases when the concentration within the sediment layer approaches its maximum permissible value ϕ_m, can be obtained by expanding (4.3) with $\psi = \phi_m - \phi$ and integrating once using the condition $F''' = 0$ at $\eta = \delta$. This results in

$$F'' = 2\tan\alpha \frac{(\phi_m - \phi_s)}{\phi_m^4}(\delta - \eta)\,\psi^2 + O(\psi^3), \tag{5.4}$$

which when integrated twice, using the estimate $\psi \leqslant \psi_\delta = \phi_m - \phi_\delta$ and the conditions $F = F' = 0$ at $\eta = 0$ and $F = F_\delta$ at $\eta = \delta$, yields

$$\delta > \left\{\frac{3\phi_m^4 F_\delta}{(\phi_m - \phi_s)(\phi_m - \phi_\delta)^2 \tan\alpha}\right\}^{\frac{1}{3}}. \tag{5.5}$$

Finally, expanding (4.10) and (4.11) in a similar manner and substituting for F_δ yields the bound

$$\delta > \left\{\frac{8(1 - \phi_m)\,\phi_m}{3(\phi_m - \phi_s)(\phi_m - \phi_\delta)\tan\alpha}\right\}^{\frac{1}{3}}, \tag{5.6}$$

which predicts that, for any angle of inclination $\alpha < \frac{1}{2}\pi$, the sediment-layer thickness should become very large when $\phi_s \rightarrow \phi_m$ and when $\phi_\delta \rightarrow \phi_m$, i.e. when $\phi_s \rightarrow 0$.

Figure 7 shows the domain of existence of a steady-state boundary-layer solution as a function of the important physical parameters ϕ_s and α. It is seen that a steady motion can exist only above a minimum angle of inclination approximately equal to 15° and that, for α larger than this minimum, the domain of steady solutions lies above a curve along which ϕ_0, the particle concentration in the sediment layer adjacent to the inclined solid surface, is at its maximum possible value. At this concentration the effective viscosity becomes infinite and all motion ceases, including sedimentation. The shape of the curve $\phi_0 = \phi_m$ further underscores the fact that, according to our analysis, a steady solution cannot exist when $\phi_s \rightarrow 0$ or as $\phi_s \rightarrow \phi_m$. This last result is understandable because, although the high particle volume fraction

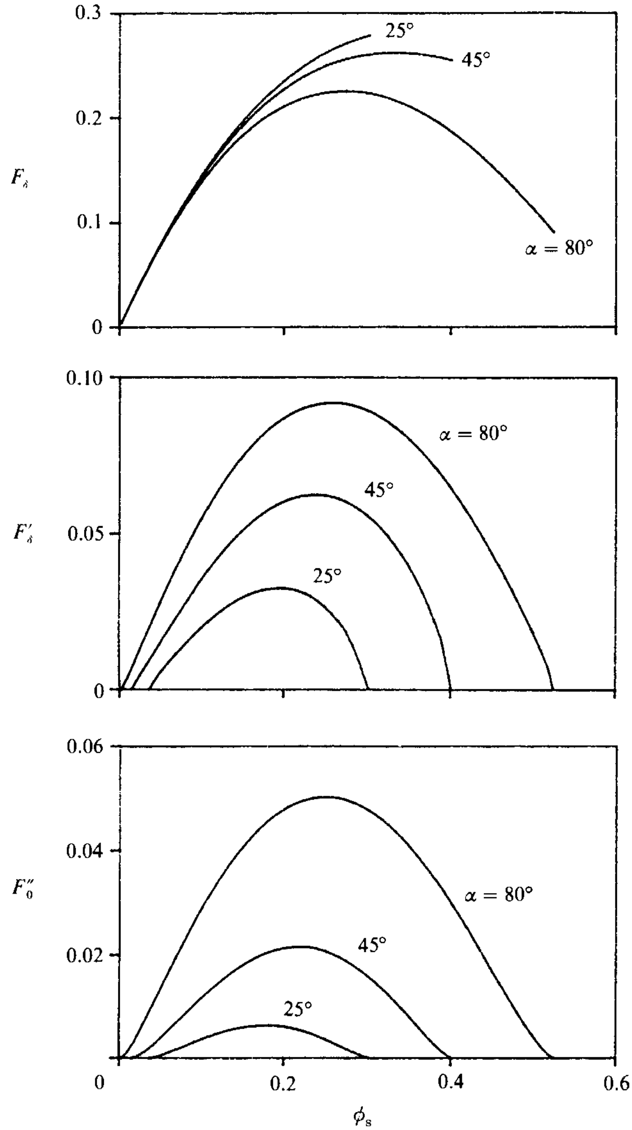

FIGURE 5. Important flow field parameters, F_δ and F'_δ at the edge of the viscous layer and F''_0 at the inclined solid surface as functions of ϕ_s and α.

in the sediment layer increases the particle diffusivity, the augmented viscosity reduces the effective shear rate and thereby the diffusion coefficient. On the other hand, the predicted rapid increase in δ as $\phi_s \rightarrow 0$ is puzzling and may well reflect the limitations of the model on which the present analysis is based.

There are of course several limitations to our model which we wish to briefly indicate. First, our analysis is based on a continuum representation according to which the sediment is viewed as an effective Newtonian fluid having an effective viscosity that depends only on the local volume fraction of the solids. Assuredly this is a gross simplification because there is ample evidence that concentrated suspensions exhibit a non-Newtonian behaviour as $\phi \rightarrow \phi_m$. Moreover, even if the Newtonian description were to be retained, it is conceivable that the quantitative

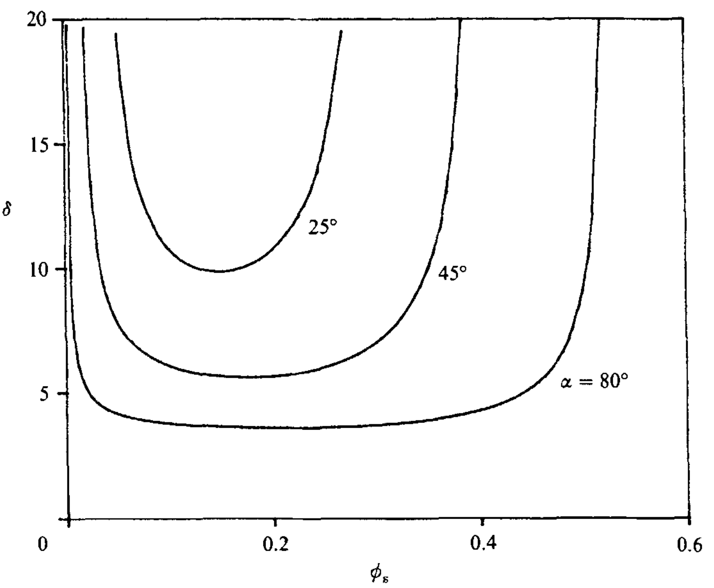

FIGURE 6. The viscous-layer thickness parameter, δ, as a function of ϕ_s and α.

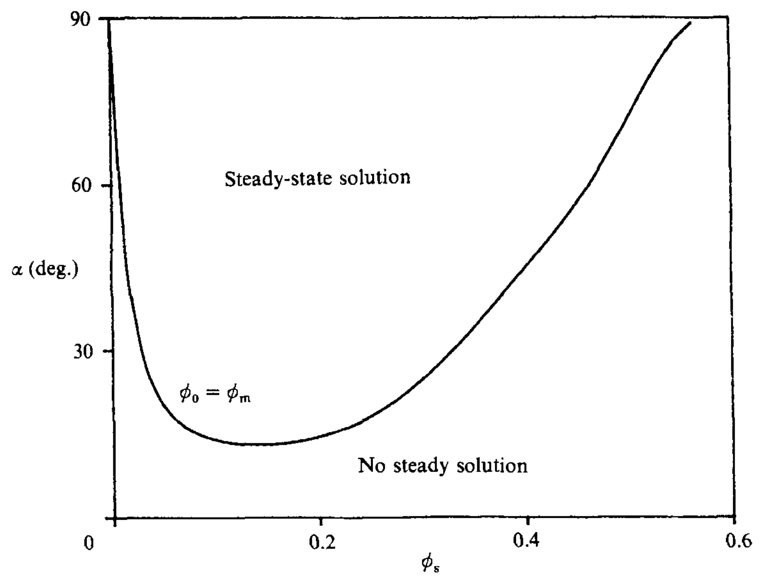

FIGURE 7. The domain of existence of a self-sustained steady-state motion of the sediment layer.

and qualitative features of our model would be significantly affected if (5.2) and (5.3) were replaced by different expressions. Finally, owing to the finite size of the particles comprising the suspension, it would be more logical to replace the no-slip condition in (2.14) by a more general boundary condition involving a slip bulk velocity proportional to the local shear rate in the fluid adjacent to the wall, or alternatively by using a position-dependent expression for the effective viscosity such that it equals the viscosity of the pure liquid as the solid surface is approached. These and other refinements of the model deserve further study.

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Appendix

An approximation for the velocity and concentration profiles can be found using the Kármán–Pohlhausen integral method (Schlichting 1968). Assume that F and ϕ have the polynomial form

$$F = C_1 \xi^2 + C_2 \xi^3 \tag{A 1}$$

and

$$\phi - \phi_s = C_3 + C_4 \xi + C_5 \xi^2, \tag{A 2}$$

with $\xi = \eta/\delta$.

Note that F satisfies (4.5). Integrating equations (4.3) and (4.4) across the boundary layer and using (4.6) and (4.7) yields

$$\delta^3 \int_0^1 (\phi - \phi_s)\,\mathrm{d}\xi = \frac{2}{9\tan\alpha}\left(\lambda \frac{\mathrm{d}^2 F}{\mathrm{d}\xi^2}\right)_{\xi=0} \tag{A 3}$$

and

$$\int_0^1 F \frac{\mathrm{d}(\phi - \phi_s)}{\mathrm{d}\xi}\,\mathrm{d}\xi + (f\phi)_{\xi=1} = 0 \tag{A 4}$$

which together with (4.7) and (4.12) constitute the approximated set of equations for the coefficients C_1 to C_5 and the boundary-layer thickness δ.

The explicit expression for δ is

$$\delta = \left\{ \frac{3(\phi_\delta + 2\phi_0 - 3\phi_s) F_\delta \beta(\phi_0)}{f(\phi_0)\,\phi_0} \left[1 - \left(1 - \frac{4}{3} \frac{\phi_0(1-\phi_0)}{(\phi_\delta + 2\phi_0 - 3\phi_s)^2 \beta(\phi_0) \tan\alpha} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{3}}, \tag{A 5}$$

where ϕ_0 is determined by the implicit equation

$$11 F_\delta(\phi_\delta - \phi_0) + \frac{7}{6} \frac{f(\phi_0)\,\phi_0\,\delta^3}{\beta(\phi_0)} + 20 \phi_\delta f(\phi_\delta) = 0. \tag{A 6}$$

A comparison of the results obtained by this approximation with those calculated by a direct numerical integration of (4.3) and (4.4) subject to (4.5), (4.7) and (4.12), shows that for large ϕ_s, i.e. when the jump at the layer edge diminishes, the two sets agree to within a few percent. As ϕ_s is reduced, the difference further diminishes until it effectively vanishes as $\phi_s \rightarrow 0$.

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